Lecture 2

Let us continue with graphing $r = \sin\theta$.

An easy way to try graphing polar equations is to sample several points

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
r	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0

Plotting these (and more if you like) and tracing out the curve gives the circle (see page 2).

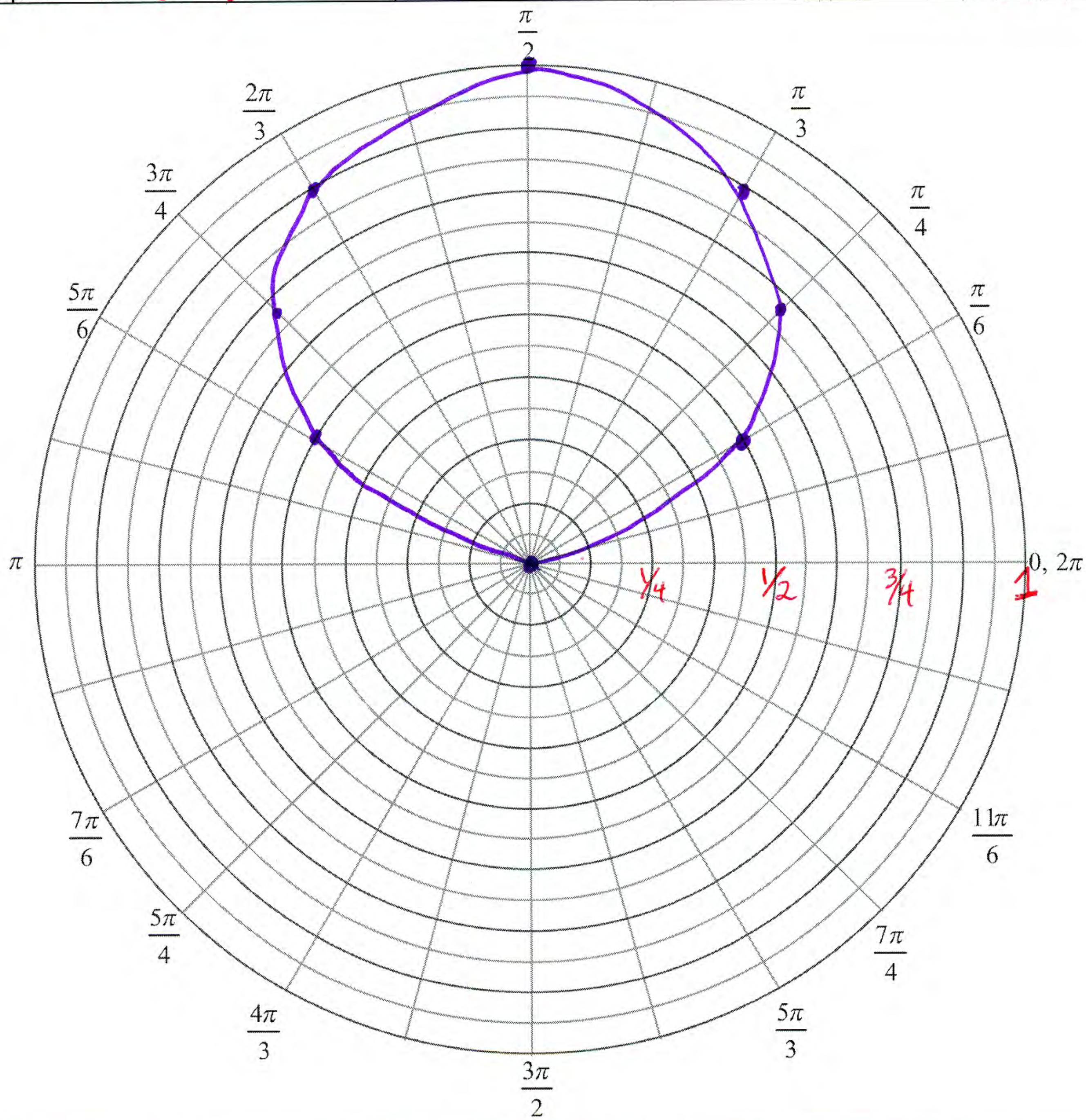
What happens if we go past $\theta = \pi$?

θ	π	$7\pi/6$	$5\pi/4$	$4\pi/3$	$3\pi/2$	$5\pi/3$	$7\pi/4$	$11\pi/6$	2π
r	0	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$	-1	$-\sqrt{3}/2$	$-\sqrt{2}/2$	$-1/2$	0

Tracing this out, we see that it just retraces the same circle!

(recall, if $r < 0$, $(r, \theta) = (|r|, \theta + \pi)$)

Name: Description: r = sin θ	Section/Page Numbers:	Problems:	Date:	Page 2-2



$\sqrt{2}/2 \approx 0.71$

$\sqrt{3}/2 \approx 0.87$

$7/8 = 0.875$

$11/16 = 0.6875$

(for reference)

Ex: Graph $r=\theta$

Sol: This is a simple, yet important graph called the spiral of Archimedes.

See page 4 for the graph.

We graph it, again, by just plotting points.

Sometimes a little more care is needed to graph these equations. For a graph $r=f(\theta)$, where $f(\theta)$ is a continuous function, an important fact to note is that r cannot switch from positive to negative (or vice-versa) without crossing 0. So, sometimes it helps to find the zeros of $f(\theta)$.

Ex: Graph $r=\cos 3\theta$.

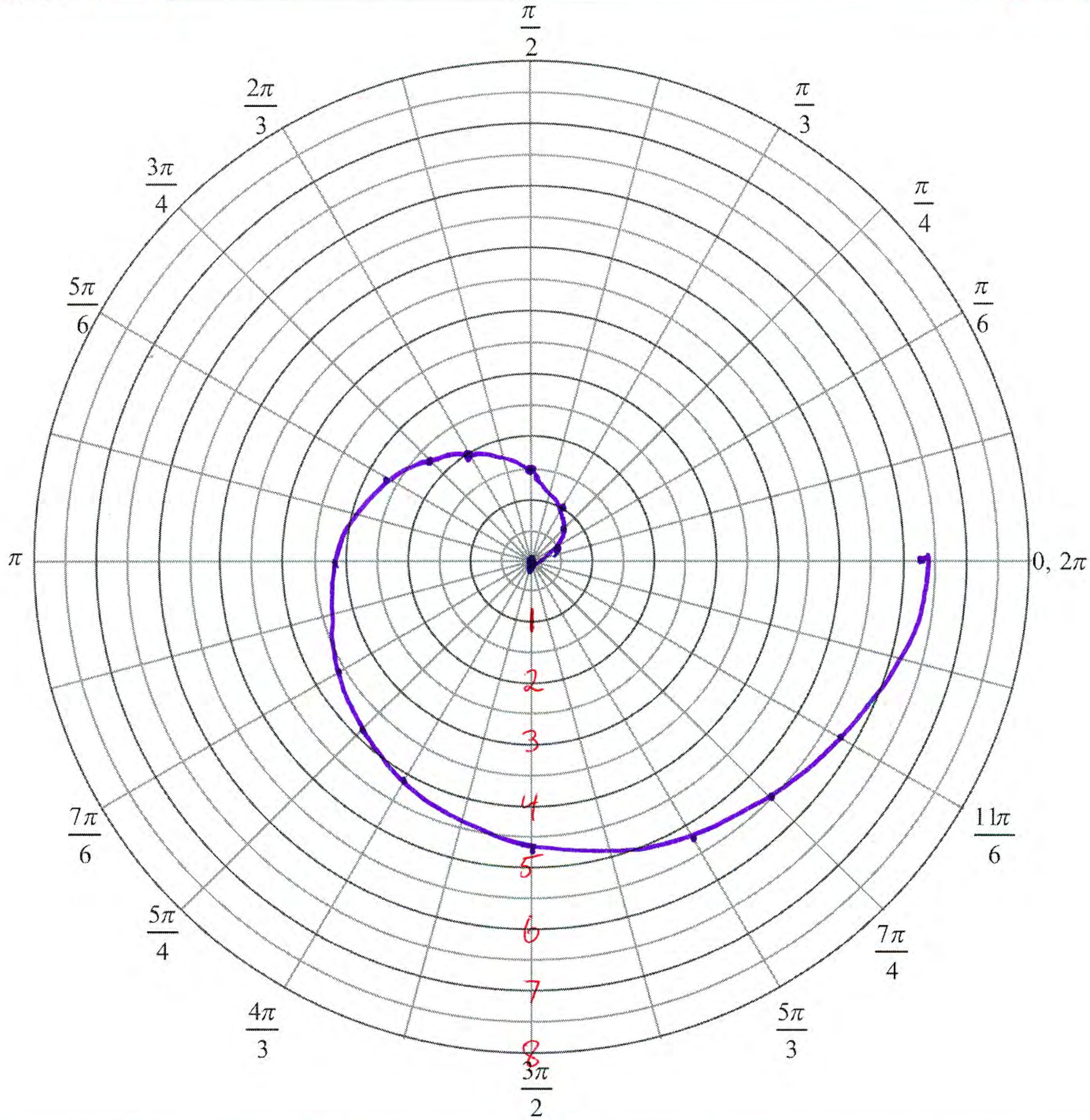
Sol: Start by finding when $\cos(3\theta)=0$.

$\cos(\theta)=0$ when $\theta=\frac{\pi}{2}+n\pi$, so $\cos(3\theta)=0$ when

$$3\theta=\frac{\pi}{2}+n\pi \Leftrightarrow \theta=\frac{\pi}{6}+\frac{n\pi}{3}=\frac{\pi}{6}+\frac{2n\pi}{6}=\frac{(2n+1)\pi}{6}$$

$$r=\cos 3\theta=0 \text{ @ } \theta=\dots,-\frac{\pi}{2},\frac{-\pi}{6},\frac{\pi}{6},\frac{\pi}{2},\dots$$

Name: Description: $r=\theta$	Section/Page Numbers:	Problems:	Date:	Page 2+4



$\pi/6 \approx 0.52$	$2\pi/3 \approx 2.09$	$7\pi/6 \approx 3.66$	$5\pi/3 \approx 5.24$
$\pi/4 \approx 0.79$	$3\pi/4 \approx 2.36$	$5\pi/4 \approx 3.93$	$7\pi/4 \approx 5.50$
$\pi/3 \approx 1.05$	$5\pi/6 \approx 2.62$	$4\pi/3 \approx 4.19$	$11\pi/6 \approx 5.76$
$\pi/2 \approx 1.57$	$\pi \approx 3.14$	$3\pi/2 \approx 4.71$	$2\pi \approx 6.28$

Let's focus on the range $[\frac{-\pi}{6}, \frac{\pi}{6}]$.

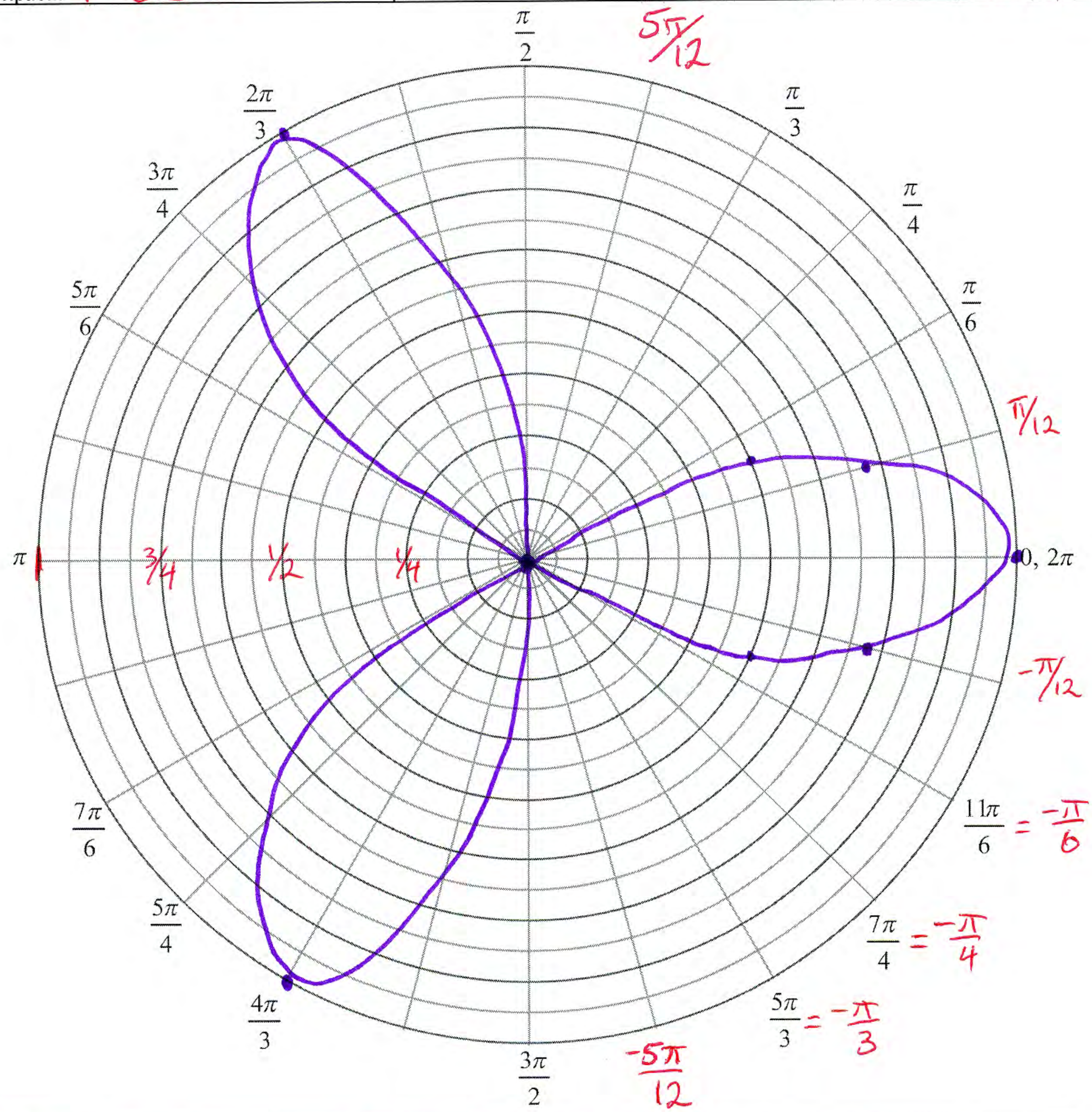
To really see what happens here, let's check several values inside this interval

3θ	$-\pi/2$	$-\pi/3$	$-\pi/4$	$-\pi/6$	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
θ	$-\pi/6$	$-\pi/9$	$-\pi/12$	$-\pi/18$	0	$\pi/18$	$\pi/12$	$\pi/9$	$\pi/6$
r	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0

If we look at the intervals $[\frac{-\pi}{2}, \frac{-\pi}{6}]$ and $[\frac{\pi}{6}, \frac{\pi}{2}]$, we will see the same pattern, but with negative r-values. If we look at the next interval of this size, $[\frac{\pi}{2}, 5\pi/6]$, we see positive r-values, and so the graph corresponds to the graph for $[\frac{-\pi}{2}, \frac{-\pi}{6}]$. The case is similar for the rest. Thus the graph is given by a <u>3-petaled rose</u>.

See page 6.

Name: Description: $r = \cos 3\theta$	Section/Page Numbers:	Problems:	Date:	Page 2-6



$\sqrt{2}/2 \approx 0.71$ $\quad 7/8 = 0.875$

$\sqrt{3}/2 \approx 0.87$ $\quad 11/16 = 0.6875$

Conic Sections in Polar Coordinates

Let $\varepsilon \geq 0$ and $d > 0$.

We will study the equation

$$(*) \quad r = \frac{d}{1 + \varepsilon \cos\theta}$$

We will break down the study of (*) into 3 parts: (i) $\varepsilon = 1$, (ii) $\varepsilon < 1$, (iii) $\varepsilon > 1$

(i) $\varepsilon = 1$

$$r = \frac{d}{1 + \cos\theta}$$

Since $\cos\theta \geq -1$, we see that $r > 0$ for $-\pi < \theta < \pi$ and undefined when $\theta = \pm\pi$.

Let's examine $0 \leq \theta < \pi$ first.

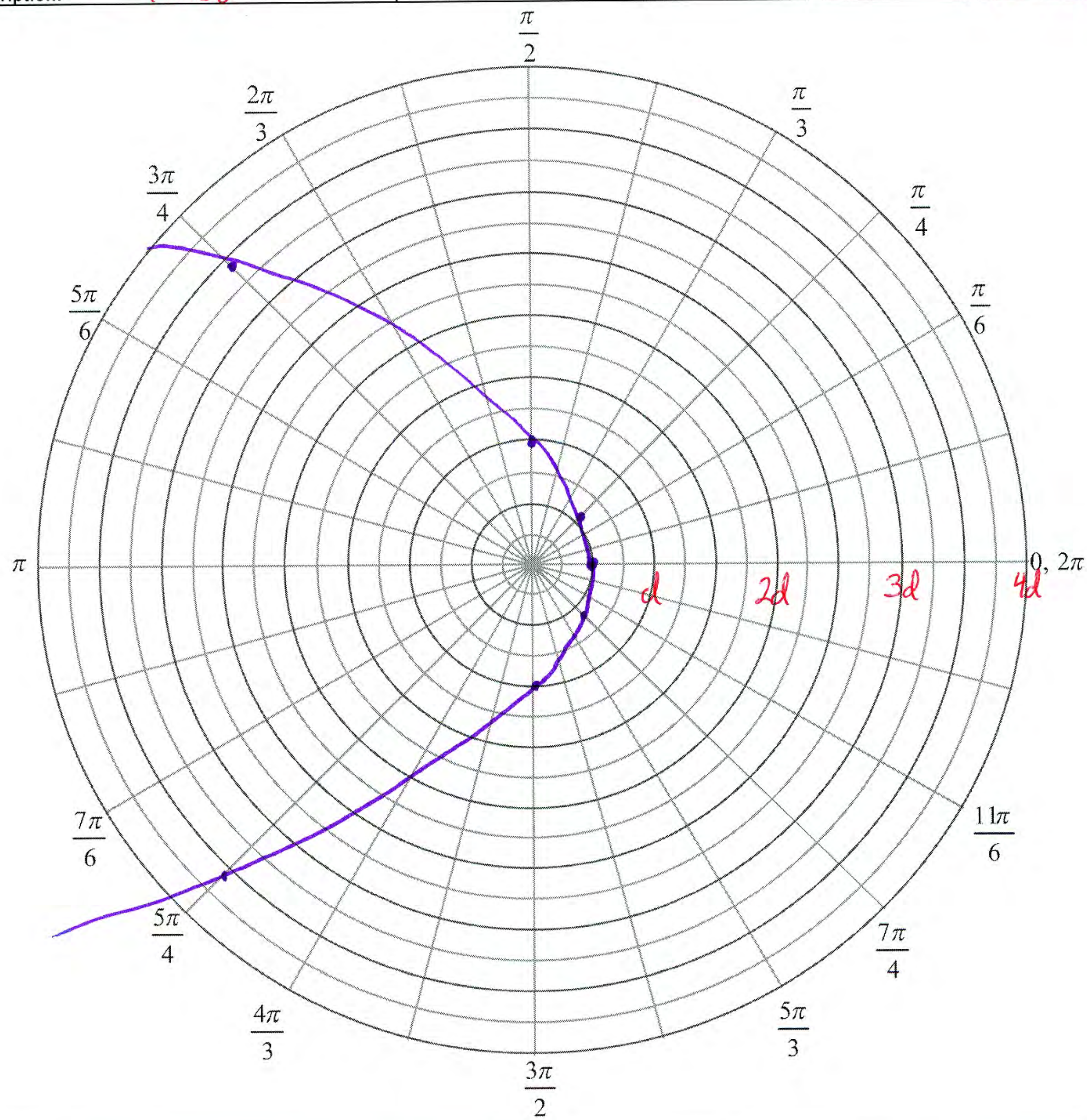
Checking the usual points gives:

θ	0	$\pi/4$	$\pi/2$	$3\pi/4$
$\cos\theta$	1	$\sqrt{2}/2$	0	$-\sqrt{2}/2$
r	$\frac{d}{2}$	$0.6d$	d	$3.4d$

Using the fact that $\cos(-\theta) = \cos(\theta)$ we know the values for $-\pi < \theta \leq 0$.

So, the graph looks like:

Name: Description: $r = \frac{d}{1+\cos\theta}$	Section/Page Numbers:	Problems:	Date:	Page 2-8



This looks like a parabola with focus at the origin! Let's verify this:

$$r = \frac{d}{1+\cos\theta}, \quad -\pi < \theta < \pi$$

$$\Rightarrow r + r\cos\theta = d$$

$r>0$

$$\Rightarrow \sqrt{x^2+y^2} + x = d \Leftrightarrow \boxed{\sqrt{x^2+y^2} = d - x}$$

(Notice $x \leq \frac{d}{2}$ here since $d = \sqrt{x^2+y^2} + x \geq x + x = 2x$.)

The red boxed equation says the distance from a point (x,y) on the curve to the origin is equal to the distance from (x,y) to the line $x=d$ $(|x-d| = d-x$ if $x \leq d)$

This is the definition of a parabola with focus at the origin and directrix $x=d$.

So

$$r = \frac{d}{1+\cos\theta}$$ is a parabola with focus $(0,0)$ and directrix $x=d$.

It might help you see this by the following

$$\sqrt{x^2+y^2} = d - x$$

square

$$\Rightarrow \cancel{x^2} + y^2 = d^2 - 2dx + \cancel{x^2}$$

$$\Rightarrow 2dx = -y^2 + d^2 \Rightarrow x = \frac{-1}{2d}y^2 + \frac{d}{2}$$

Lecture 3

Before doing $\varepsilon > 1$ & $\varepsilon < 1$, let's do the following:

$$r = \frac{d}{1+\varepsilon\cos\theta} \iff r + \varepsilon r\cos\theta = d$$

$$\Rightarrow \pm\sqrt{x^2+y^2} + \varepsilon x = d$$

$$\Rightarrow \pm\sqrt{x^2+y^2} = d - \varepsilon x$$

$$\text{(square)} \Rightarrow x^2 + y^2 = d^2 - 2\varepsilon dx + \varepsilon^2 x^2$$

$$\Rightarrow (1-\varepsilon^2)x^2 + 2\varepsilon dx + y^2 = d^2$$

$$\Rightarrow x^2 + \frac{2\varepsilon d}{1-\varepsilon^2}x + \frac{y^2}{(1-\varepsilon^2)} = \frac{d^2}{1-\varepsilon^2}$$

$$\left(\begin{array}{c}\text{complete}\\ \text{the square}\end{array}\right) \Rightarrow x^2 + \frac{2\varepsilon d}{1-\varepsilon^2}x + \frac{\varepsilon^2 d^2}{(1-\varepsilon^2)^2} + \frac{y^2}{1-\varepsilon^2} = \frac{d^2}{1-\varepsilon^2} + \frac{\varepsilon^2 d^2}{(1-\varepsilon^2)^2}$$

$$\Rightarrow \left(x + \frac{\varepsilon d}{1-\varepsilon^2}\right)^2 + \frac{y^2}{1-\varepsilon^2} = \frac{d^2}{1-\varepsilon^2}\cdot\frac{1-\cancel{\varepsilon^2}}{1-\varepsilon^2} + \frac{\varepsilon^2 \cancel{d^2}}{(\cancel{1}-\varepsilon^2)^2} = \left(\frac{d}{1-\varepsilon^2}\right)^2$$

$$\Rightarrow \boxed{\frac{\left(x+\frac{\varepsilon d}{1-\varepsilon^2}\right)^2}{\left(\frac{d}{1-\varepsilon^2}\right)^2} + \frac{y^2}{\left(\frac{d^2}{1-\varepsilon^2}\right)} = 1} \quad (**)$$